DIRAC ELECTRON IN THE ELECTRIC DIPOLE FIELD

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Abstract

Dirac equation for the finite dipole potential is solved by the method of the join of the asymptotics. The formulas for the near continuum state energy term of a relativistic electron-dipole system are obtained analytically. Two cases are considered: $Z < 137$ and $Z > 137$

Dirac equation for $Z > 137$ is solved by usual method of cut-off potential at small distances.

We would like to discuss the problem of the Dirac equation for an electron in an electric dipole field which is closely related to the problem of exotic heavy quarkonium states too. We mean the problem of the motion of light valence and vacuum quarks in the color field of heavy quark-antiquark pair. The motion of an electron under the influence of an electric dipole one of the classical problem of nonrelativistic quantum mechanics. Energy eigenvalues of such system was found by many authors see e.g. [1, 2, 3]. Several authors have calculated the critical value of the dipole moment at which energy term reaches the boundary of continuum [1, 2].

Solution of corresponding relativistic problem is more difficult mathematical problem, since variables of the Dirac equation can't be separated at any orthogonal system of coordinates. Near continuum energy state of the relativistic electron, moving in the field of finite electric dipole, we shall find by the method of the join of the asymptotics which successfully was applied to solution of two-center Coulomb problem for the Dirac equation (with same signs of charges)[6, 7]. Previously this problem was considered in our paper [8].

In the present paper we also use asymptotical separability of variables at large distances in the relativistic two-center Coulomb problem and additionally discuss the nonrelativistic limit of our results. Then the squared Dirac equation takes a form of the Schrodinger equation at which variables can be separated. Let us consider relativistic electron moving in the field of a finite electric dipole which is composed of charges $+Z\alpha$ and $-Z\alpha$ are separated by distance R. The potential of this system is given by

$$V = \frac{Z\alpha}{r_1} - \frac{Z\alpha}{r_2} \tag{1}$$

where r_i is the distance from i th charge, $\alpha^{-1} = 137$ (the system of units $\hbar = m = c = 1$ is used here and below). The motion of this electron is described by Dirac equation which can be written as

$$i\frac{\partial\psi}{\partial t} = H\psi \ , \tag{2}$$

where $H = \vec{\alpha}\vec{p} + \beta + V$ is the Dirac Hamiltonian, α and β are the Dirac matrices.

In the spinor representation stationar squared equation can be written in the components ψ_1 and ψ_2 of a byspinor $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ as in [5]

$$[(\varepsilon - V)^2 + \triangle + i\vec{\sigma}\vec{\nabla}V - 1]\psi_1 \tag{3}$$

$$[(\varepsilon - V)^2 + \triangle + i\vec{\sigma}\vec{\nabla}V - 1]\psi_2 \tag{4}$$

where ε is energy of an electron, $\vec{\sigma}$ are standard Pauli matrices.

At large distances from dipole we can neglect by terms V^2 and $i\vec{\sigma}\vec{\bigtriangledown}V$ with respect to V hence for each component we have equation

$$[\triangle - 2\varepsilon V + \varepsilon^2 - 1]\psi = 0 \tag{5}$$

Eq.(5) is the Schrodinger equation with potential εV and with energy $-\frac{1}{2}(\varepsilon^2 - 1)$ So, at large distances the wave function has separated form

$$\psi = \frac{U(\xi)}{(\xi^2-1)^{1/2}} \frac{V(\eta)}{(1-\eta^2)^{1/2}}\, exp(im\varphi) \tag{6}$$

where ξ, η, φ are the spheroidal coordinates which are defined by $\xi = (r_1 + r_2)R^{-1}$, $\eta = (r_1 - r_2)R^{-1}$, $\varphi = arctg(\frac{y}{x})$, where y and x are the cartesian coordinates, R is the distance between charges, $1 < \xi < \infty\,,\ \ -1 \leq \eta \leq 1\,,\ \ 0 \leq \varphi \leq 2\pi$

The radial and angular equations become

$$U''(\xi) \ + \ [-p^2 \ + \ \frac{A}{1-\xi^2} \ + \ \frac{1-m^2}{(1-\xi^2)^2}]U(\xi) \ = \ 0 \tag{7}$$

$$V''(\eta) \ + \ [-p^2 \ + \ \frac{D\eta \ - \ A}{1 \ - \ \eta^2} \ + \ \frac{1 \ - \ m^2}{(1 \ - \ \eta^2)^2}]V(\eta) \ = \ 0 \tag{8}$$

where $p^2 = -\frac{R^2}{4}(\varepsilon^2 - 1)$, $D = 2 \mid \varepsilon \mid RZ\alpha$, A is the constant of separation.

The boundary conditions are

$$U(1) = 0,\ U(\xi)_{\xi \longrightarrow \infty} \longrightarrow 0,\ V(\pm 1) = \ 0 \tag{9}$$

Asymptotes of eq.(7) for large ξ is [1] $U(\xi) \ = \ (p\xi)^{1/2}K_{i\nu}(p\xi)$, where $K_{i\nu}$ is the McDonald function, $\nu^2 = A - \frac{1}{4}$. Solution of eq.(8) in the vicinity of $\eta = 0$ was also found in ref.[1].

This asymptotes has the form

$$V(\eta) \ = \ [z'(\eta)]^{-\frac{1}{2}}\, Ai(-D^{\frac{1}{3}}z) \tag{10}$$

where Ai is the Airey function, $z(\eta) = [\frac{3}{2}\int_0^\eta \eta^{1/2}(1-\eta^2)^{-1/2}d\eta]^{\frac{2}{3}}$ and $z'(\eta) = \frac{dz}{d\eta}$.

Thus for large $p\xi$ and for small η asymptotes of wave function can be written in the form(for $m = 0$)

$$\psi = [\frac{\pi}{\nu sh\pi\nu}]^{\frac{1}{2}}(p\xi)^{\frac{1}{2}}(\xi^2-1)^{-\frac{1}{2}}Sin(\nu ln[\frac{2}{p\xi} + arg\gamma(1+i\nu)]) \tag{11}$$

Here we have used asymptotes of function $K_{i\nu}(x)$ for $x \longrightarrow 0$ [6]

$$K_{i\nu}(x) \longrightarrow [\frac{\pi}{\nu sh\pi\nu}]^{\frac{1}{2}}(p\xi)^{\frac{1}{2}}sin\nu ln[\frac{2}{x} + arg\gamma(1+i\nu)]$$

and the asymptotes of Airey function [15] $Ai(z) \longrightarrow c_1 - c_2 z$ for $z \longrightarrow 0$, where $c_1 = 0.355, c_2 = 0.259$, and the relation $z'(\eta) \approx 1$ for $\eta \longrightarrow 0$.

Following by Popov [6] as an asymptotes of wave function at small distances we will take the product of relativistic one-center wave functions. Here we must consider two cases: $Z < 137$ and $Z > 137$.

Let $Z < 137$. In this case asymptotics (at small distances from center) is given by [5]

$$\phi_1 = r^{-1+\gamma} \tag{12}$$

where $\gamma = (1 - Z^2\alpha^2)^{\frac{1}{2}}$

Thus asymptotes of two center wave function at small distances from dipole can be written as follows:

$$\psi_1 \sim (r_1 r_2)^{\gamma-1} = (\xi^2 - \eta^2)^{\gamma-1} \tag{13}$$

Let now $Z > 137$. In this case energy becomes imaginary ("fall to the center") [9, 10]. In order to solve the Dirac equation in this case we must cut-off the Coulomb potential at small distances [11, 12].

General form of the cut-off potential is [13, 14]

$$V(r) = \begin{cases} \frac{Z\alpha}{r}\,, & for\ \ r > b \\ \frac{Z\alpha}{b} f(r)\,, & for\ \ 0 < r < b \end{cases}\,,$$

where b is the radius of the nuclei.

For $f(r) = 1$ (surface distribution of charge) asymptotes of wave function has the form[13]

$$\phi_2 \sim r \tag{14}$$

Hence for the case of $Z > 137$ asymptotics of two center wave function at small distances from dipole can be written in the form

$$\psi \sim r_1 r_2 \sim \xi^2 - \eta^2. \tag{15}$$

So in general case one can write for the asymptotics of wave function at small distances from dipole (for $R << 1$)

$$\psi_1 \sim (\xi^2 - \eta^2)^{\beta}, \tag{16}$$

where

$$\beta = \begin{cases} \gamma - 1\,, & for\ \ Z > 137 \\ 1\,, & for\ \ Z > 137 \end{cases}\,,$$

For small η

$$\psi_1 \sim \xi^{2\beta} \tag{17}$$

Since $|\varepsilon| \approx 1$ asymptotes of wave function at small and at large distances join with other in the region $1 << \xi << \frac{1}{p}$. Equating logarithmic derivatives of functions (17) and (11) and taking into account that in this region $ln\frac{2}{p\xi} \approx ln\frac{2}{p}$ we have

$$\frac{1}{2} + \nu ctg[\nu ln\frac{2}{p} + arg\Gamma(1+i\nu)] = -2\beta \tag{18}$$

From this expression we get

$$p = exp(\frac{1}{\nu}arg\Gamma(1+i\nu) - \frac{1}{\nu}arcctg[-\frac{1+4\beta}{2\nu}] - 2ln2) \tag{19}$$

In order to find the $\varepsilon(R)$ from (19) we must know the parametr ν. Here we use results of ref.[1] where dipole moment was found as a function of ν. In application to our problem for ground state this formula can be written as

$$D = \frac{\Gamma^4(\frac{1}{4})}{32\pi}[1 + \frac{92}{3\pi}(4p^2 + 6\nu^2 - 1)] \tag{20}$$

where $D = 2 \mid \varepsilon \mid ZR\alpha$. From this equation we find ν :

$$\nu = [\frac{8\pi^2}{\Gamma^4(\frac{1}{4})}[D - 2 \mid \varepsilon \mid ZR\alpha - D_{cr}] - \frac{R^2}{6}(\varepsilon^2 - 1)]^{\frac{1}{2}} \tag{21}$$

where $D_{cr} = \frac{\Gamma^4(\frac{1}{4})}{32\pi}[1 - \frac{2}{3\pi}]$.

is the critical value of dipole moment at which term $\varepsilon(R)$ reaches the boundary of continuum. Taking into account relation between p and ε we find

$$\varepsilon_{\pm} = \pm[1 - \frac{2}{R^2}exp(\omega(\nu, Z))] \tag{22}$$

where

$$\omega(\nu, Z) = \frac{2}{\nu}arg\Gamma(1+i\nu) - \frac{2}{\nu}arcctg[-\frac{1+4\beta}{2\nu}] - 2ln2 \tag{23}$$

Thus for $Z < 137$ we have transcendental equation for $\varepsilon(R)$

$$\varepsilon_{\pm} = \pm 1 \mp \frac{2}{R^2}exp[\frac{2}{\nu}arg\Gamma(1+i\nu) - \frac{2}{\nu}arcctg\frac{3-4\gamma}{2\nu} - 2ln2] \tag{24}$$

where ν defined by (21).

For $Z < 137\sqrt{7}/4$ eq.(24) can be solved by iterations (since iteration procedure converges only for $3 - 4\gamma < 0$) So for the first iteration we have

$$\varepsilon_{\pm} = \pm 1 \mp \frac{2}{R^2}exp[-\frac{\Gamma^2(\frac{1}{4})}{\pi(4Z\alpha R - 2D_{cr})^{\frac{1}{2}}}arcctg\frac{\Gamma^2(\frac{1}{4})(3-4\gamma)}{4\pi(4Z\alpha R - 2D_{cr})^{\frac{1}{2}}} \tag{25}$$

Formula (25) is the near continuum state energy term of an electron in the electric dipole field. As well known nonrelativistic limit gets near the continuum. Since for near continuum

states $4ZaR - 2D_{cr} \sim 0$, $arcctg \longrightarrow \pi$. Thus in order to obtain nonrelativistic limit we must do following replacements: 1)$\gamma \longrightarrow 1$, 2) $arcctg \longrightarrow \pi$; as a result we have

$$\varepsilon_{\pm} = \pm 1 \mp \frac{2}{R^2} exp[-\frac{\Gamma^2(\frac{1}{4}}{\pi(4Z\alpha R - 2D_{cr})^{\frac{1}{2}}}] \tag{26}$$

This formula for $\varepsilon_{\pm}$ coincides with known formula from [2] for the nonrelativistic electron-dipole system. By expanding arccotangent into series in formula (25) we calculate the correction to eq.(26)

$$\varepsilon \approx \frac{2}{R^2} exp[-\frac{\Gamma^2(\frac{1}{4}}{\pi(4Z\alpha R - 2D_{cr})^{\frac{1}{2}}} - \frac{4}{3 - 4\gamma}] \tag{27}$$

From this formula it is easy to see that nonrelativistic limit take place for $(4Z\alpha R - 2D_{cr})^{\frac{1}{2}} << 0.8$. We note that this nonrelativistic limit can be also obtained directly from eq.(19). For $Z > 137$ we have equation

$$\varepsilon_{\pm} = \pm 1 \mp \frac{2}{R^2} exp[\frac{2}{\nu} arg\Gamma(1 + i\nu) - \frac{2}{\nu} arcctg \frac{5}{2\nu} - 2ln2 - \frac{2\pi}{\nu}] \tag{28}$$

Solving this equation by iterations for the first iteration we have

$$\varepsilon_{\pm} \approx \pm 1 \mp \frac{2}{R^2} exp[-\frac{\Gamma^2(\frac{1}{4})}{\pi(4Z\alpha R - 2D_{cr})^{\frac{1}{2}}} (arcctg \frac{5\Gamma^2(\frac{1}{4})}{4\pi(4Z\alpha R - 2D_{cr})^{\frac{1}{2}} - \pi})] \tag{29}$$

Derived analytical formulas have to be useful for further numerical calculations in nonasymptotical region.

We are planning to generalize our results for the Coulomb plus confinement potential case for the calculations of the properties of the exotic quarkonium states.

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